Dynamics of Intelligent Systems

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Abstract

Recent advances in nonlinear dynamics demonstrate a remarkable complexity of patterns outside of equilibrium, which are derived from simple basic laws of physics. There has been identified a class of mathematical models providing a variety of such patterns in the form of static, periodic or chaotic attractors. These models appeared to be so general that they predict not only physical, but also biological, economical and social patterns of behavior. Such a phenomenological reductionism may suggest that, on the dynamical level of description, there is no difference between a solar system, a swarm of insects, and a stock market. However, this conclusion is wrong for a very simple reason: Even primitive living species possess additional non-Newtonian properties, which are not included in the laws of Newtonian or statistical mechanics. These properties follow from a privileged ability of living species to possess the self-image (the concept introduced in mathematical psychology). In this paper we consider the existence of the self-image as a postulate to be added to classical physics for modelling behavior of living systems. We show that self-image can be incorporated into the mathematical formalism of nonlinear dynamics which evolve in the probability space. We will demonstrate that one of the basic invariants of living systems is their ability to predict future, and that will be associated with intelligence.

I.   Introduction

Modelling of life can be performed on many different levels of description. In this paper we will be concerned with geometrical invariants of biosignatures representing prints of behavioral patterns. One of the most remarkable patterns in biology is the formation of specie aggregation as an evolutionary advantageous state, in which members derive benefits of protection, mate choice, and centralized information, balanced by the costs of limiting resources. Consisting of individual members, aggregations nevertheless function as an integrated whole, displaying a complex set of behaviors not possible at the level of the individual organism. Aggregation occurs at all sizes from bacteria to whales, from groups of 10 to 10 million. Therefore, in the world of bacteria, biosignatures should be associated with configuration of aggregations. Operationally, aggregations fit into two classes: Those that self-organize and those that form in response to external objectives (light, food). Depending upon that, they may have a variety of geometrical forms (milling in which individual members circle about an unoccupied core, polarity without a leader, distinct shape whose topology varies to suit the tasks, etc.). Each of such configurations can be associated with a legend which explains its evolutionary advantage [2].

The most powerful modeling tool for analysis of biological patterns is based upon fundamental paradigm of nonlinear dynamics called attractor. Attractor is a stable dissipative structure which does not depend (at least, within a certain basis) upon the initial conditions. Due to this property, the whole history of evolution prior to attraction becomes irrelevant, and that represents a great advantage for information processing, and in particular-for pattern recognition.

The mathematical approach to pattern formation is based upon the theory of active systems (both natural and artificial) which is described by a system of PDE [1].

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$$\dot{a}_i = g_i(\{a\}) + \sum_j \beta_{ij}(\nabla a_j)^2 + \sum_{ij} D_{ij}\nabla^2 a_j, \quad 1,2\ldots n \tag{1}$$

where $\{a\} = a_1\ldots a_n$ are state variables, g is a multi-extremal function, and β_{ij}, D_{ij} are constants.

The properties of the solutions to Eq. (1) in terms of the type of the attractor depend upon a certain dimensionless control parameter R (such as Reynolds or Reyleigh numbers). This dependence may lose its uniqueness at certain critical points when $R=R_{cr}$, and the solution becomes linearly unstable. Because of the richness of postinstability structures, many different stable patterns (both deterministic or chaotic) may appear when $R=R_{cr}$. These structures include effects of fluid dynamics, nonlinear diffusion, chemical kinetics, etc., and its solutions can form such patterns as Bernard cells, Taylor vortices, trigger and spiral waves, traveling pulses, etc. Surprisingly, the same equations are exploited for simulating biological patterns such as transplantation and regeneration in hydra, compartment formation in drosophila, mammalian coat markings, pigment patterns on mollusk shell etc. However, such an "universality" of Eq. (1) immediately disqualifies it as a tool for the detection of life since it does not suggest any mechanisms for life-nonlife discrimination. Therefore it is not a coincidence that the main success in simulating biological patterns using Eq. (1) is associated with the morphogenesis, i.e., with structures during the growth of an organism rather than with a collective behavior of swarms or colonies. Indeed, in contradistinction to a set of physical particles which interact via flows of energies, a living species interact via flows of information and that is not captured by Eq. (1). The flows of information are produced and processed by a signaling system whose complexity is ranging from interactions between single molecular to interactions between species in ecological systems, and that may include receptors, transducers, enzymes, and diffusable second messengers, etc.[2] . In order to incorporate the signaling phenomena into the process of patterns formation on the same

level of description, the following phenomenological approach based upon ideas proposed in [3] will be developed.

2. **Reflexive dynamics**

In contradistinction to physical systems, a biological system, from the viewpoint of nonlinear dynamics can be considered as a multi-body system (with "bodies" represented by cells) which is interconnected via information flows. Since these flows as well as responses to them may be distorted, delayed, or incomplete, the motion of each cell becomes stochastic, and it can be simulated by a controlled random walk. This random walk is caused not by an external noise (as in the case of a physical particle) but rather by an internal effort (a "free will") triggered by the signaling system. Physically it is represented by an ordered sequence of runs, pausing and tumbles.

One of the main challenges in modelling living systems is to distinguish a random walk of a physical origin (for instance, Brownian motions) from those of a biological origin and that will constitute the starting point of the proposed approach. As conjectured in [3] , the biological random walk must be nonlinear. Indeed, any stochastic Markov process can be described by linear Fokker-Planck equation (or its discretized version); only those types of processes have been observed in the inanimate world. However, all such processes always converge to a stable (ergodic or periodic) state, i.e., to the states of a lower complexity and higher entropy. At the same time, the evolution of living systems is directed toward a higher level of complexity if complexity is associated with a number of structural variations. The simplest way to mimic such a tendency is to incorporate a nonlinearity into the random walk; then the probability evolution will attain the features of the Burgers equation: the formation and dissipation of shock waves initiated by small shallow wave disturbances. As a result, the evolution never "dies": it produces new different configurations which are accompanied by increase or decrease of entropy (the

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decrease takes place during formation of shock waves, the increase-during their dissipation). In other words, the evolution can be directed "against the second law of thermodynamics" by forming patterns outside of equilibrium.

In order to elucidate both the physical and the biological aspects of the proposed model, let us start with a one-dimensional random walk:

$$x_{t+\tau} = x_t + hSgn[R + \mu], h = Const, \tau = Const. \tag{2}$$

where h and τ are the space (along x) and time steps respectively; R is a random function taking values from -1 to 1 with equal probability, μ is a control parameter while $|\mu| \leq 1/2$. (Physical implementations of this model are discussed in the Appendix).

Eq. (1) describes motion in actual physical space. But since this motion is irregular, it is more convenient to turn to the probability space:

$$f_{t+\tau,x} = pf_{t,x-h} + (1-p)f_{t,x+h}; \quad f_{t+\tau} = f(t+\tau,x) \tag{3}$$

where f(x,t) is the probability that the moving particle occupies the point x at the instant t, and the transition probability

$$p = \frac{1}{2} + \mu, \quad 0 \leq p \leq 1 \tag{4}$$

It should be noticed that at the reflecting boundary $\mu = \pm\frac{1}{2}$.

It is well known[4,5] that if the system interacts with the external world, i.e.,

$$\mu = \mu(x), \text{ and therefore, } p = p(x)$$

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the solution to Eq. (3) subject to the reflecting boundary conditions converges to a stable stochastic attractor. However, if

$$\mu = \mu(f), \text{ and therefore, } p = p(f) \tag{5}$$

Eq. (3) becomes nonlinear, and Eq. (2) is coupled with Eq. (3) via the feedback (4).

From the physical viewpoint, the system (2), (3) can be compared with the Langevin equation which is coupled with the corresponding Fokker-Planck equation such that the stochastic force is fully defined by the current probability distributions, while the diffusion coefficient is fully defined by the stochastic force. The process described by this system is Markovian since future still depends only upon present, but not past. However, now present includes not only values of the state variable, but also its probability distribution, and that leads to nonlinear evolution of random walk.

From the mathematical viewpoint, Eq. (2) simulates probabilities while Eq. (3) manipulates by their values. The connection between these equations is the following: if Eq. (2) is run independently many times and a statistical analysis of these solutions is performed, then the calculated probability will evolve according to Eq. (3).

From the biological viewpoint, Eqs. (2) and (3) represent the same subject: a simplest living specie, or, using terminology introduced by Leibniz, a monad. Eq. (2) simulates its motor dynamics, i.e., actual motion in physical space, while Eq. (3) can be associated with mental dynamics describing information flows in the probability space. Such an interpretation [3] was evoked by the concept of reflection in psychology. Reflection is traditionally understood as the human ability to take the position of an observer in relation to one's own thoughts[11]. In other words, the reflection is a self-awareness via the interaction with the "image of the self." In terms of the phenomenological formalism proposed above, Eq. (3) represents the probabilistic "image" of the dynamical system (1). If this system "possesses" its own image, then it can predict, for instance, future expected values of its parameters, and by interacting with

the image, change the expectations if they are not consistent with the objective. In this context, Eq. (1) simulates acting, and Eq. (2) simulates "thinking." Their interaction can be implemented by incorporating probabilities, its functions and functionals into the control parameter μ (see Eq. (5)). From the cognitive viewpoint, μ implements the self-awareness associated with the amount of information which the system possesses about its self-image.

In general Eq. (3) is representable in the form (1), and therefore, it possesses a variety of different complex patterns outside of equilibrium. However, in contradistinction to Eq. (1), Eq. (3) simulate patterns in the probability space, i.e., in the space of the mental dynamics so that the corresponding actual motions in physical space are described by nonlinear random walks (2). Due to that, a specie is not locked up in a certain pattern of behavior: it still can perform a variety of motions, and only the statistics of these motions is constrained by this pattern.

3. **Emerging Self-Organization**.

We will start the analysis of the coupled motor-mental dynamics with Eqs. (2) and (3) where:

$$p = Sin^2\left(af_{t,x} + \beta\right), \mu = p - \frac{1}{2}, a, \beta = Const, \quad f = f(x,t) \tag{6}$$

i.e.,

$$x_{t+\tau} = x_t + hSgn\left[R + Sin^2\left(af_{t,x} + \beta\right) + \frac{1}{2}\right] \tag{7}$$

$$f_{t+\tau,x} = f_{t,x-h}Sin^2\left(af_{t,x} + \beta\right) + f_{t,x+h}Cos^2\left(af_{t,x} + \beta\right) \tag{8}$$

Here a and β are constant weights, or control parameters.

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In order to illustrate the fundamental nonlinear effects, we will analyze the behavior of special critical points by assuming that

$$a = \frac{5\pi}{12}, \beta = -\frac{\pi}{6} \quad \text{and}$$

$$f_0 = f(t = 0) = \begin{cases} f_0^{(1)} = \frac{1}{5} \ at \ x = -\ell \\ f_0^{(2)} = \frac{4}{5} \ at \ x = \ell \\ f_0^{(3)} = 0 \ \ \text{otherwise} \end{cases} \tag{9}$$

Then the solution to Eq. (8) will consist of two waves starting from the points $x = -\ell$ and $x = \ell$, traveling toward each other with the constant speed $\upsilon = h / \tau$, and transporting the values $f_0^{(1)}$ and $f_0^{(2)}$, respectively, i.e.,

$$f = f_0^{(1)}\left(-\ell + \frac{h}{\tau}n\right) + f_0^{(2)}\left(\ell - \frac{h}{\tau}n\right), n = 0,1,\dots\frac{\ell}{h}, \tag{10}$$

where n is the number of time-steps.

At $n = 1/h$, the waves confluence into one solitary wave at $x = 0$:

$$f = \begin{cases} 1 \ at \ x = 0 \\ 0 \ \ \text{Otherwise} \end{cases} \quad at \quad t = n\tau = \frac{\ell}{h}\tau \tag{11}$$

This process represents a discrete version of formation and confluence of shock waves, and it is characterized by a decrease of the Shannon entropy from

$$H(0) = -\frac{1}{5}\log_2\frac{1}{5} - \frac{4}{5}\log_2\frac{4}{5} > 0 \quad to \ H(n\tau) = 0 \tag{12}$$

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$$H(0) = -\frac{1}{5} \ell og_2 \frac{1}{5} - \frac{4}{5} \ell og_2 \frac{4}{5} > 0 \quad to \ H(n\tau) = 0 \qquad (12)$$

However, the solitary wave (11) is not stationary. Indeed, as follows from Eq. (8), the solution at $t = (n+1)\tau$ splits into two equal values:

$$f_{(n+1)\tau} = \begin{cases} 1/2 \ at \ x = \ell \pm h \\ 0 \qquad \text{Otherwise} \end{cases} \qquad (13)$$

The process (13) can be identified as a discrete version of diffusion during which the entropy increases again from

$$H(n\tau) = 0 \quad to \ H[(n+1)\tau] = -\ell og_2 \frac{1}{2} = 1 \qquad (14)$$

The further evolutionary steps $t \geq (n+2)\tau$ will include both diffusion and wave effects, and therefore, the solution will endlessly display more and more sophisticated patterns of behavior in the probability space. The corresponding solutions to Eq. (7) represent samples of the stochastic process (8) in the form of non-linear random walks in actual physical space.

Thus, the solutions to coupled motor-mental dynamics simulate emerging self-organization which can start spontaneously. At this level of description, such an effect is triggered by instability rather than by a specific objective. In other words, the model represents a "brainless" life. However, it serves well to the global objective of each living system: the survival. Indeed, it is a well established fact in biology [6] that marginal instability makes behaviors of living system more flexible and therefore, more adaptable to changing environment.

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The model (2), (3) is easily generalizable to three-dimensional motions:

$$x_{t+\tau}^{(i)} = x_t^{(i)} + h_i Sgn\left[(R + \mu_i)\right] \quad i = 1,2,3 \tag{15}$$

$$f_{t+\tau,x^{(i)}} = \sum_{i=1}^{3}\left[p_i f_{t,x^{(i)}-h_i} + (1 - p_i) f_{t,x^{(i)}+h_i}\right] \tag{16}$$

Here $x^{(1)}, x^{(2)}, x^{(3)}$ are the space coordinates, and $f = f\left(x^{(1)}, x^{(2)}, x^{(3)}, t\right)$ is the joint probability that the specie occupies the point $x^{(1)}, x^{(2)}, x^{(3)}$ at the instant t.

As in the one-dimensional case, here

$$p_i = \frac{1}{2} + \mu_i, \quad 1,2,3$$

In particular, one can assume that

$$p_i = Sin^2\left(a_i f_{t,x} + \beta\right), \quad i = 1,2,3 \tag{17}$$

It should be noticed that the nonlinear random walks (15) in all three directions are coupled by means of joint probability f via the control parameters μ_i.

From the mathematical viewpoint, the model of mental dynamics (8) links to the Burger's equation in a sense that its pattern formation outside of equlibrium is based upon the balance between dissipation and shock waves.

In general, this model can be enriched with the Belousov-Zhabotinskii effects by slight modification of random walk (7):

$$x_{t+\tau} = x_t + \frac{1}{2}h\{1 - Sgn[R + \mu]\}Sgn[R + \mu] \tag{18}$$

which now includes the third choice for the specie: to remain at rest with the probability

$$q = \frac{1}{2} + \mu, \tag{19}$$

The corresponding version of Eq. (8) reads:

$$f_{t+\tau,x} = qf_{t,x} + pf_{t,x-h} + (1 - q - p)f_{t,x+h}, \quad 0 \le p,q \le 1 \tag{20}$$

If $q = q(f)$ and $p = p(f)$, one arrives at the discretized version of the combined Burger-Belousov-Zhabotinskii equation which possesses a variety of new complex patterns outside of equilibrium, and that increases the adaptability of species to environmental changes. One should recall that Belousov-Zhabotinskii equation was already exploited for studying patterns formation in biology [1]. However, these patterns dwell in physical space; in contradistinction to that, Eqs. (18) and (20) simulate patterns in the probability space, i.e., in the space of the mental dynamics so that the corresponding actual motions in physical space are described by nonlinear random walks (7) and (18) respectively. Again, due to that, a specie is not locked up in a certain pattern of behavior: it still can perform a variety of motions, and only the statistics of these motions is constrained by this pattern. It should be emphasized that such a "twist" is based upon the concept of reflection, i.e., the existence of the self-image.

4. __Feedback from expected future__.

The feedback (6) from mental to motor dynamics was expressed via the current probability distribution $f = f(x,t)$. In general, one can include in (6) memories $f = f(x, t - \tau)$ and non-local effects $f = f(x \pm h, t)$. In all these cases, the mental dynamics evolves independently upon the motor dynamics. This property allows living systems to predict future by running the self-image faster than real time, and then correct (if necessary) the motor dynamics (7) via the feedback from expected future. Actually such a privilege of living systems represents the basic component of the concept of intelligence.

Let us now show how this phenomenon can be implemented in the model of motor-mental dynamics. For this purpose one has to modify the feedback (6) as follows:

$$p' = Sin^2\left(\alpha' f_{t_0 + \tau, x} + \beta'\right), \qquad \mu' = p' - \frac{1}{2}; \; \alpha, \beta = Const \qquad (21)$$

where $f_{t_0 + \tau}$ is found from Eq. (8).

It should be emphasized that $f_{t_0 + \tau}$ is the expected distribution for $t > t_0$ since it is not yet effected by the new feedback (21). The real future distribution $f'_{t_0 + \tau}$ is found from the modified mental dynamics:

$$f'_{t_0 + \tau, x} = f'_{t, x - h} p' + f'_{t, x + h}(1 - p'), \; t > t_0 \qquad (22)$$

in which p' is defined by Eq. (21). Eq. (22) can be rewritten in the explicit form for the case when starting from $t_0 + \tau$, the feedback (6) was replaced by the feedback (21):

$$f'_{t_0+\tau,x} = f'_{t,x-h}Sin^2\left\{\alpha'\left[f'_{t,x-h}Sin^2\left(\alpha f'_{t,x} + \beta\right) + f'_{t,x+h}Cos^2\left(\alpha f'_{t,x} + \beta\right)\right] + \beta'\right\} +$$

$$+f'_{t,x+h}Cos^2\left\{\alpha\left[f'_{t,x-h}Sin^2\left(\alpha f'_{t,x} + \beta\right) + f'_{t,x+h}Cos^2\left(\alpha f'_{t,x} + \beta\right)\right] + \beta\right\} \tag{23}$$

The process described by Eq. (23) is still Markovian despite the fact that present here is correlated with future: indeed, as was demonstrated above, the (expected) future is uniquely defined by present (see Eq. (8)). However, the process (23) is more sophisticated than those described by Eq. (8) and it can be useful for systems with objective since then the feedback from expected future plays the role of a gradient (in the probabilistic space) which guides the evolution of the motor dynamics.

The feedback (21) can be presented in a more general form:

$$p' = Sin^2 F\left(f_{t_0+n\tau}\right), \quad n = 1,2\ldots etc., \quad \mu = p' - \frac{1}{2} \tag{24}$$

where F is an arbitrary function, and $f_{t_0+n\tau}$ is an expected distribution at $t = t_0 + n\tau$ found from the equation:

$$f_{t+\tau,x} = p_0 f_{t,x-h} + \left(1 - p_0\right)f_{t,x+h}, \quad p_0 = p' \ at \ t = t_0 \tag{25}$$

5. **Systems with global objective**.

As shown in the previous sections, the solutions to Eq. (8), or to its generalized versions, Eqs. (15), (22) and (25) can simulate emerging temporary self-organization which is characterized by a decrease of the Shannon entropy (see Eq. (12)). In this section we will discuss a link between this phenomenon and global objectives of a living system.

First of all, as follows from the relation between finite differences E and derivatives D:

$$E_h = e^{hD} \tag{26}$$

the continuous representation of Eqs. (8), (15), (22) and (25) contains the derivatives of all orders:

$$A_1 \frac{\partial f}{\partial t} + A_2 \frac{\partial^2 f}{\partial t^2} + \ldots = \beta_1 f + \beta_2 \frac{\partial f}{\partial x} + \beta_3 \frac{\partial^2 f}{\partial x^2} + \beta_4 \frac{\partial^3 f}{\partial x^3} + \ldots etc. \tag{27}$$

while all the coefficients depend upon f and the control parameters $\partial, \beta, \partial', \beta', etc$ (see Eqs. (6), (17), (21), and (24)).

Therefore, the solutions to Eq. (27) may include the effects of the Burger's equation [7] $\left(A_1 = 1, \ B_2 = f, \ B_3 = \text{Const, the rest } A_{i,} B_1 = 0 \right)$ the Korteweg-deVries equation[8] $\left(A_1 = 1, \ B_2 = f, \ B_4 = 1, \text{ the rest } A_1 B_i = 0 \right)$ and the Belousov-Zhabotinskii equation [1] $\left(A_1 = 1, B_1 \neq 0, B_3 = \text{Const the rest } A_i, B_i = 0 \right)$ which are, respectively: formation and diffusion of shock waves, formation of trains of solutions, and formation of trigger waves. The transitions from one pattern to another formally can be achieved by an appropriate change of the control parameters α, β, etc. In order to illustrate that, let us turn to the matrix representation of Eq. (3).

$$\pi(t + \tau) = \pi(t)P \tag{28}$$

Here the vector $\pi = \pi_1, \pi_2 \ldots \pi_N$ represents the probability distribution $f(x,t)$ at the points $x = 1,2,\ldots N$ so that

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$$\pi_i(t) = f(x_i, t), \quad i = 1, 2, \ldots N, \quad \sum_{i=1}^{N} \pi_i = 1 \tag{29}$$

For the reflective boundary conditions at x=1 and x=N, the matrix P has the following forms [5] :

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1-p & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & p & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 1-p & p & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \tag{30}$$

If

$$p = Const \tag{31}$$

Eq. (25) has a closed form solution [5] which tends to a stationary distribution for large number of steps

$$n \gg N \tag{32}$$

$$\pi_k(n \gg N) \rightarrow \frac{\frac{p}{1-p} - 1}{\left(\frac{p}{1-p}\right)^N - 1} \left(\frac{p}{1-p}\right)^{k-1} \tag{33}$$

and

$$p^n = \begin{pmatrix} \pi_1 & \cdots & \pi_N \\ \pi_1 & \cdots & \pi_N \\ \vdots & & \vdots \\ \pi_1 & & \pi_N \end{pmatrix} \qquad \text{for } n \gg N \tag{34}$$

We will pose now the following problem: store a set of m stationary stochastic processes given by vectors of their probability invariants

$$I^{(i)} = I_1^{(i)}, I_2^{(i)}, \dots I^{(i)}; \quad i = 1, 2, \dots m \tag{35}$$

(These invariants can be represented, for instance, by expectation E, variance and higher moments) in such a way that when presented with any of the process $\overset{*(j)}{\pi}$ out of the set of M processes

$$\dot{\pi}^{(j)} = \dot{\pi}_1^{(j)}, \dot{\pi}_2^{(j)} \dots \dot{\pi}_N^{(j)}, \quad j = 1, 2, \dots M \tag{36}$$

Eq. (28) converges to one of the stochastic processes (35).

The performance

$$\pi \to I \tag{37}$$

represents correspondence between two classes of patterns, i.e., a hetero-associative memory on a high level of abstraction. Indeed, each process in (37) stores an infinite number of different pattern of behaviors which, however, are characterized by the same sequence of invariants (35) and (36), respectively thereby representing a decision making strategy. Considering a living system as a decision-maker, one can give the following interpretation of the mapping (37):

Classical artificial intelligence as well as artificial neural networks are effective in a deterministic and repetitive world, but faced with uncertainties and unpredictabilities, both of them fail. At the same time, many natural and social phenomena exhibit some degree of regularity only on a higher level of abstraction, i.e., in terms of some invariants.

For instance, each particular realization of a stochastic process can be unpredictable in details, but the whole ensemble of these realizations i.e., "the big picture" preserves the probability invariants (expectation, moments, information, etc.) and therefore, predictable in terms of behavior "in general."

Hence, if the strategy of the decision-maker is characterized by a pattern $\dot{\pi}^{(i)}$ from (36), and starting from t=0, the external information becomes unavailable, he should change its strategy from pattern $\dot{\pi}^{(i)}$ to the corresponding pattern from (35), and that can be associated with a decision based upon "common sense." It is implied that the attracting strategies I are sufficiently "safe," i.e., they minimize the risk taken by the decision-maker in case of an uncertain external world.

We will illustrate the approach by the simplest case when there are m attracting stochastic processes stored, and each of them is characterized by only one invariant-the expectation $E^{(i)}$, $i = 1, 2, \ldots m$, while the rest invariants are not specified.

The first step in the implementation of the mapping (37) is to find an appropriate feedback. Departing from Eqs. (21) and (24), we will seek the nonlinear feedback in the form:

$$p = Sin^2\left(\sum_{i=2}^{N-2} \alpha_i \pi_i + \alpha_1\right), \quad \sum_{i=1}^{N} \pi_i = 1 \tag{38}$$

$$\pi_1 = 0, \quad \pi_N = 0 \tag{39}$$

Eq. (38) introduces N-2 weights $\alpha_i (i = 1, 2, \ldots N - 2)$ to implement the mapping (37), while Eq. (39) enforces reflecting boundaries.

Substituting Eq. (38) into Eq. (33) one obtains:

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$$\pi_k = \frac{\tan^2\left(\sum_{i=2}^{N-2}\alpha_j\pi_i + \alpha_1\right)-1}{\tan^{2N}\left(\sum_{i=2}^{N-2}\alpha_i\pi_i + \alpha_1\right)-1}\tan^{k-1}\left(\sum_{i=2}^{N-2}\alpha_i\pi_i + \alpha_1\right), \quad k=1,2,\ldots N << n \qquad (40)$$

One should notice that Eq. (33) (which was derived under asspumption p=Const) is still valid for the case (38) since starting with n>>N, the stochastic process is supposed to be stationary (provided by an appropriate choice of the weights α_i), and therefore $\pi_i = Const$ in Eq. (38)

The existence of m stationary stochastic processes with expectations $E^{(i)}(i=1,2,\ldots m)$ requires that N-2 weight coefficients α_i satisfy the following m=N-2 equations:

$$E^{(i)} = \sum_{K=1}^{N-3}\left[k\frac{\tan^2\left(\sum_{j=1}^{N}\alpha_j\pi_j^{(i)} + \alpha_0\right)-1}{\tan^{2n}\left(\sum_{j=1}^{N}\alpha_j\pi_j^{(i)} + \alpha_0\right)-1}\tan^{k-1}\left(\sum_{j=1}^{N}\alpha_j\pi_j^{(i)} + \alpha_0\right)-1\right], \quad i=1,2,\ldots m \qquad (41)$$

Here $E^{(i)}$ are expectations of the stochastic processes (35) to be stored, which are given and $\pi_j^{(i)}$ are expectations of the stochastic processes (36) characterizing the original states of the system (which are also given). In order to solve this system in a dynamical way, one can apply the simplest version of the back-propagation strategy by minimizing the "energy":

$$E = \sum_{i=1}^{N-2}\left\{E^{(i)} - \sum\left[k\frac{\tan^2\left(\sum_{j=1}^{N-3}\alpha_j\pi_j^{(i)} + \alpha_1\right)-1}{\tan^{2N}\left(\sum_{j=1}^{N-3}\alpha_j\pi_j^{(i)} + \alpha_1\right)-1}\tan^{k-1}\left(\sum_{j=1}^{N-3}\alpha_j\pi_j^{(i)} + \alpha_1\right)-1\right]\right\} \rightarrow \min \qquad (42)$$

i.e., by using the following recurrent relationships:

$$\alpha_i^{(j+1)} = \alpha_i^{(j)} - \lambda_i^2 \frac{E_j - E_{j-1}}{\alpha_i^{(j)} - \alpha_i^{(j-1)}}, \quad \lambda_i^2 = Const, \quad i = 1, 2, \ldots N - 2 \tag{43}$$

where

$$E_j = E\left(\alpha^{(j)}\right) \tag{44}$$

Here $\alpha^{(j)}$, and E_j are the jth approximations of the corresponding values of α and E.

It should be recalled that Eqs. (43) represent a gradient dynamical system, and therefore, the recurrent procedure in (43) always converges. However, since the nonlinearity of Eq. (42) (with respect to α_j) is not quadratic, the solution to Eq. (43) can be trapped in a local minima. Therefore, the recurrent procedure should be repeated several times starting from different initial values of $\alpha^{(0)}$, and then the lowest minima of E has to be chosen as the solution.

Thus, the dynamical system (2), (3) represented in the equivalent form:

$$x_{t+\tau} = x_t + h \, Sgn(R + \mu), \tag{45}$$

$$\pi(t + \tau) = \pi(t)P \tag{46}$$

where the matrix P is expressed by Eq. (30), and

$$p = Sin^{(2)}\left(\sum_{j=2}^{N-3} \alpha_j \pi_j + \alpha_1\right), \quad \alpha_j = Const, \quad \pi_1 = 0, \quad \pi_N = 0, \quad \sum_{j=1}^{N} \pi_j = 1, \tag{47}$$

$$\mu = p - \frac{1}{2} \qquad\qquad (48)$$

possesses the following property:

If the initial value $x_{t=0}$ in Eq. (45) is drawn from a stochastic process $\pi^{(j)}$ (see Eqs. (36)), then the solution to Eqs. (45) and (46) will approach (for n>>N) a new stochastic process which is characterized by a prescribed expectation $E^{(j)}$ (see Eq. (32)). This attracting stochastic process can be associated with the global objective of the underlying living system.

We will now make three comments concerning the mapping (37) introduced above.

Firstly, strictly speaking not all of the stochastic processes from Eq. (35) are true attractors: some of them can be repellers. Indeed, the corresponding weights α_j were found from statical rather than dynamical conditions (see Eqs. (41)), and therefore, the stability of these stochastic processes were not established. The situation here is similar to those in neural nets where some of the equilibrium (or fixed) points are stable, and some of them are not. In terms of Markov chains (see Eq. (46)), the possibility that some of the stochastic processes (35) are unstable (and therefore, will never be approached) follows from the nonlinearity (47). It should be recalled that classical Markov chains are linear, and all the processes with reflecting boundaries coverge to stable stochastic processes.

Secondly, Eqs. (33), (34),(40) etc., include the condition that n>>N, i.e., that the number of time-steps is significantly larger than the number of space-steps N. Actually this condition can be specified if one evokes a well known result from the Markov chain theory[5] which quantifies Eq. (33):

$$\pi_{jk}^{(n)} = \frac{\frac{p}{1-p}-1}{\left(\frac{p}{1-p}\right)^{N}-1}\left(\frac{p}{1-p}\right)^{k-1} + \frac{2^{n+1}p^{1+\frac{1}{2}(n-j+k)}(1-p)^{\frac{1}{2}(n+j-k)}}{N}\sum_{r=1}^{N-1}S_r \dots \quad (49)$$

where

$$S_r = \frac{Cos^n \frac{\pi r}{N}\left\{\frac{\pi r j}{N}-\left(\frac{1-p}{p}\right)^{\frac{1}{2}}Sin\frac{\pi r(j-1)}{N}\right\}\left\{Sin\frac{\pi r k}{N}-\left(\frac{1-p}{p}\right)^{\frac{1}{2}}Sin\frac{\pi r(k-1)}{N}\right\}}{1-2[p(1-p)]^{\frac{1}{2}}Cos\frac{\pi r}{N}} \quad (50)$$

As $n \to \infty$, Eq. (49) tends to Eq. (33), while the second term in Eq. (49) allows one to evaluate n such that this term can be ignored in comparison to the first term.

Eq. (50) (as well as Eq. (33)) was derived under assumption that p=Const. However, it is still valid for the case (38) as long as the second term in Eq. (49) can be ignored since then the stochastic process is stationary, and therefore $\pi_i = Const$ in Eq. (38). Obviously, this conclusion is true only if the stochastic process in (35) is stable.

Thirdly, if the attracting stochastic processes in Eq. (37) must be specified not only by the expectations (35), but also by higher moments, the additional (to Eq. (41)) constraints

$$\sum_{j=1}^{N-3}k\pi_j^r = M_r^{(i)}, \; (r=2,3,\dots r_x), \; i=1,2,\dots m \quad (51)$$

should be imposed upon the weights α_j.

Here $M_r^{(r)}$ are the specified moments for the ith stochastic process, π_j are the probabilities expressed by Eq. (40), and m is the number of the attracting stochastic processes, while

$$m = \frac{N-2}{r} \quad (52)$$

The number of the prescribed moments m can be increased if the feedback (38) is generalized to the following form:

$$p = Sin^2\left(\alpha_1 + \sum_{i=2}^{N-2}\alpha_i\pi_i + \sum_{i,j=2}^{N-2}\alpha_{ij}\pi_i\pi_j + \dots etc.\right)$$

6.   **Systems with hidden identity**

A living system may have a global objective which is different from those described in the previous section. Indeed, let us consider a biological or social system which is in the state of a prey-predator game. Then it may be beneficial for such a

system to make its behavior as unpredictable as possible. For a physical system that would mean the maximization of the Shannon entropy subject to the constraints imposed upon its motion:

$$H = -\sum_{i=1}^{N} \pi_i \, log \pi_i \rightarrow \max, \quad \sum_{i=1}^{N} \pi_i = 1, \; etc.$$

(53)

But a living system can do better than that: it can mislead its adversaries by hiding its identity.

The simplest way to do that is to make the feedback from mental to motor dynamics chaotic. For that purpose, let us turn to Eq. (2) and rewrite it in the following form:

$$x_{t+\tau} = x_t + h \, Sgn\left[R + \frac{1}{2}\mu + \frac{1}{2}\left(\mu' - \frac{1}{2}\right) \right]$$

(54)

where the bias

$$\mu = Sin^2 \sum_{i=1}^{N} \alpha_i f_{t,x_i} - \frac{1}{2}, \quad -\frac{1}{2} \leq \mu \leq \frac{1}{2}$$

(55)

$$\mu'_{t+\tau} = 4\mu_t(1-\mu_t), \quad 0 \leq \mu' \leq 1$$

(56)

Eq. (55) represents the deterministic components of the non-linear feedback, and Eq. (56) – its chaotic component. Indeed, Eq. (56) is known as a logistic map which leads to chaotic time series.

Now the transition probability in Eq. (3), with reference to Eq. (4) becomes:

$$p_t = \frac{1}{2}\left(Sin^2 \sum_{i=1}^{N} \alpha_i f_{t,x_i} + \mu'_t + \frac{1}{2} \right), \quad 0 \leq p \leq 1$$

(57)

Since it includes the chaotic component $\frac{1}{2}\mu'$, the solution to Eq. (3) attains chaotic features, and therefore, the probability distribution $f(t,x)$ becomes unpredictable.

Thus, the motor dynamics (54) is now run by the mental dynamics

$$f_{t+\tau,x} = p_t f_{t,x-h} + (1-p_t)f_{t,x+h} \tag{58}$$

via the feedback (57) which includes the nonlinear deterministic component defined via the bias (55) and the chaotic component defined by the bias (56). The last component which implement the hiding of the identity by making the probability distribution $f(x,t)$ unpredictable can be associated with a deception dynamics.

7.   **<u>Systems with local objective</u>**

In many real life situations, a living system does not know, or cannot formulate its global objective. Instead, it can formulate local, i.e., a one-time-step-ahead objective. We will start with the simplest case: a predator-prey pursuit. We will assume that both the predator and prey possess not only the image of the self, but the image of the adversary as well. In terms of the three-dimensional model (15), (16). The pursuit can be formulated as follows:

$$x_{t+\tau}^{(i)} = x_1^{(i)} + h_i Sgn(R + \mu_i^{(1)}), \quad i = 1,2,3; \tag{59}$$

$$f_{t+\tau,x^i}^{(1)} = \sum_{i=1}^{3}\left[p_i f_{t,x^{(i)}-h}^{(1)} + (1-p_i)f_{t,x^{(i)}-h_i}\right], \quad i=1,2,3; \tag{60}$$

$$\tilde{f}_{t+\tau,y^{(i)}}^{(2)} = \sum_{i=1}^{3}\left[\tilde{q}_i \tilde{f}_{t,y^{(i)}-h}^{(2)} + (1-\tilde{q}_i)f_{t,y^{(i)}+h_i}^{(2)}\right], \quad i=1,2,3 \tag{61}$$

$$y^{(i)}_{t+\tau} = y^{(i)}_t + h_i Sgn\left(R + \mu^{(2)}_i\right), \quad i = 1,2,3; \tag{62}$$

$$f^{(2)}_{t+\tau,y^{(i)}} = \sum_{i=1}^{3}\left[q_i f^{(2)}_{t,y^{(i)}-h_i} + (1-q_i)f^{(2)}_{t,y^{(i)}+h_i}\right], \quad i = 1,2,3; \tag{63}$$

$$\tilde{f}^{(1)}_{t+\tau,x^i} = \sum_{i=1}^{3}\left[\tilde{p}_i \tilde{f}^{(1)}_{x^{(i)}-h_i} + (1-\tilde{p}_i)\tilde{f}^{(1)}_{x^{(i)}+h_i}\right], \quad i = 1,2,3; \tag{64}$$

Here Eq. (59) simulates the motor dynamics of the predator, i.e., a random walk in three-dimensional space. Eq. (62) describes the predator's mental dynamics, i.e., evolution of the probability $f^{(1)}\left(x^{(1)},x^{(2)},x^{(3)},t\right)$ where $x^{(i)}$ denote the predator's position, $p_1 p_2$ and p_3 are the transition probabilities which depend upon $f_{x_j} = f\left(t,x_j\right)$:

$$p_i = \frac{1}{2} + \mu^{(1)}_i = Sin^2 \sum_{j=1}^{N-1}\left(\alpha^{(i)}_j f^{(i)}_{x_j} + \alpha^{(i)}_1\right), \quad i = 1,2,3, \quad \sum_{j=1}^{N} f^{(1)}_{x_j} = 1 \tag{65}$$

where $\alpha^{(1)}_j$ are constant weights to be found.

Eqs. (62) and (63) simulate the motor-mental dynamics of the prey, where $y^{(i)}$ denote the prey's positions in space, and $f^{(2)}\left(y^{(1)},y^{(2)},y^{(3)},t\right)$ is the corresponding probability: q_1, q_2 and q_3 are the transition probabilities

$$q_1 = \frac{1}{2} + \mu^{(2)}_i = Sin^2 \sum_{j=1}^{N-1}\left(\beta^{(i)}_j f^{(2)}_{y_i} + \beta^{(i)}_1\right), \quad i = 1,2,3, \quad \sum_{j=1}^{N} f^{(2)}_{y_i} = 1 \tag{66}$$

Finally, Eqs. (61) and (64) simulate mental images of the adversaries: $\tilde{f}^{(2)}\left(\tilde{y}^{(1)},\tilde{y}^{(2)},\tilde{y}^{(3)},t\right)$, and $\tilde{q}_i$ represent the prey's images in the "mind" of the predator, and $\tilde{f}^{(1)}\left(\tilde{x}^{(1)},\tilde{x}^{(2)},\tilde{x}^{(3)},t\right)$ and $\tilde{p}_i$ represent the predator's images in the "mind" of the prey.

If the predator and the prey never met before, the best strategy for them is to assume that

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$$\tilde{q}_i = (1 - p_i), \quad \tilde{p}_i = (1 - q_i) \tag{67}$$

i.e., to consider the adversary as an extreme opposite to the self.

At this point, Eqs. (59), (60) and (62), (63) are coupled only in pairs, while Eqs. (61) and (64) are decoupled.

Now we will introduce the objectives of the pursuit: the predator objective is to minimize the distance between the prey and himself during the next n steps, and the prey's objective is to maximize the same distance.

The distance after n steps is expressed as:

$$E = \sum_{k=0}^{n} \sum_{i=1}^{3} \left(x_k^{(i)} - y_k^{(i)} \right)^2 \tag{68}$$

The only way to optimize it is to manipulate by the weights α_i and β_i in Eqs. (65) and (66) using the strategy of the gradient descent approach (see Eqs. (42) and (43)). However, here this strategy can not be applied in a direct way since neither the predator, nor the prey know their actual future positions $x^{(i)}$ and $y^{(i)}$. Therefore, these positions have to be predicted based upon their images. The images can be represented by expectations, modes or medians of the corresponding probability distributions. For instance, in case of expectations, the distance (68) is replaced by:

$$\tilde{E} = \sum_{k=0}^{n} \sum_{i=1}^{3} \left(\hat{x}_k^{(i)} - \hat{y}_k^{(i)} \right)^2 \tag{69}$$

where $\qquad \hat{x}_k^{(i)} = \sum x_k^{(i)} f^{(1)}, \quad \hat{y}_k^{(i)} = \sum y_k^{(i)} f^{(2)} \tag{70}$

Then the predator's and the prey's images of the same objective are, respectively:

$$\tilde{E}_1 = \sum_{k=0}^{n} \sum_{i=1}^{3} \left(\sum x_k^{(i)} f^{(1)} - \sum \tilde{y}_k^{(i)} \tilde{f}^{(2)} \right)^2 \tag{71}$$

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$$E_2 = \sum_{k=0}^{n} \sum_{i=1}^{3} \left(\sum \tilde{x}_k^{(i)} \tilde{f}^{(1)} - \sum y_k^{(i)} f^{(2)} \right)^2 \tag{72}$$

These images are different since neither the predator, nor the prey knows the actual probabilities $f^{(2)}$ and $f^{(1)}$ of their adversasries, and they replace them by the images $\tilde{f}^{(2)}$ and $\tilde{f}^{(1)}$, respectively (see Eqs. (65), (66) and (67). Now the strategy of the predator follows from the gradient descent minimization:

$$a_{i(1)}^{(j+1)} = \alpha_{i(1)}^{(j)} - \lambda_1^2 \frac{\tilde{E}_1^{(k)} - \tilde{E}_1^{(j-1)}}{\alpha_{i(1)}^{j} - \alpha_{i(1)}^{(j-1)}}, \quad \lambda_1 = Const. \tag{73}$$

$$\beta_{i(1)}^{(j+1)} = \beta_{i(1)}^{j} - \lambda_1^2 \frac{\tilde{E}_1^{(j)} - \tilde{E}_1^{(j-1)}}{\beta_{i(1)}^{j} - \beta_{i(1)}^{(j-1)}}, \quad \lambda_1 = Const. \tag{74}$$

Similarly, the strategy of the prey follows from the gradient descent maximization

$$\alpha_i^{(j+1)} = \alpha_i^{j} + \lambda_2^2 \frac{\tilde{E}_2^{(j)} - \tilde{E}_2^{(j-1)}}{\alpha_i^{j} - \alpha_i^{(j-1)}}, \quad \lambda_2 = Const. \tag{75}$$

$$\beta_i^{(j+1)} = \beta_i^{(j)} + \lambda_2^2 \frac{\tilde{E}_2^{(j)} - \tilde{E}_2^{(j-1)}}{\beta_i^{(j)} - \beta_i^{(j-1)}}, \tag{76}$$

Thus, prior to each move, the predator and prey find the optimal weights α_i and β_i from Eqs. (73) – (76), plug them into Eqs. (59) – (64) via Eq. (65), and then make the next ("optimal") step.

There are four comments to be made concerning the model of pursuit. First of all the system (59) – (64) is now fully interconnected via the objectives (71), (72) by means

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of Eqs. (73) – (76) and (65), (66). In particular, that means that the stochastic process (60) and (63) are correlated. But it does not necessarily mean that there exists a joint probability function $\tilde{f}(\{x\},\{y\})$ for which $f^{(1)}$ *and* $f^{(2)}$ are the conditional probabilities. Indeed, as shown in [8,9] the stochastic processes (60) and (61) are entangled in a sense that there is no such a transformation of coordinates $\{x\},\{y\}$ which would decouple them.

Secondly, each specie exploits the probabilistic images of the self and its adversary to predict future positions, and to make the best available move, and this remarkable property which is a privilege of living systems, can be associated with intelligence.

Thirdly, success of the pursuit depends upon the degree of superiority of the predators' mental capacity over those of the prey if the mental capacity is measured by the speed of learning, i.e., by finding the correct values of the weights α and β from the gradient descent (73), (74).

Fourthly, in the pursuit model, each specie can demonstrate intelligence not only via the correct prediction of the future moves, but also by making misleading moves based upon desception dynamics (56) if the feedbacks (65) and (66) are modified to the form (57).

Finally, as follows from that model, the successful pursuit can be associated with the catching of a prey by the predator, and that depends upon how well the predator predicts the prey's moves. The power to predict starts with Eq. (67) when the predator selects the transition probabilities for his image of the prey. The ability to make this simple and universal choice mimics the so called innate properties conferred by all those elements with which the specie was born. However, the choice (67), in general, may not be good enough: it does not include the specific characteristic of the prey. That is why the very first "acquaintance" between the adversaries may end in a failure on the part of the

predator. But suppose that the predator managed to catch the prey. Then he can "record" the values of the actual objective (68) and its image (71). Therefore the difference

$$|\Delta E_1| = |E - \tilde{E}_1| \tag{77}$$

will serve as a measure of the mismatch between the image and reality. Based upon this difference, the predator can correct his prey's image by appropriate change of $\tilde{q}_i$ in Eq. (67). Actually he should minimize the difference (77) as a function of $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ where:

$$\tilde{q}_i^1 = Sin^2\left(\tilde{\alpha}_i \tilde{f}^{(2)} + \tilde{\beta}_i\right), \quad i = 1, 2, 3 \tag{78}$$

i.e.,

$$\tilde{\alpha}_i^{(j+1)} = \tilde{\alpha}_i^{(j)} - \lambda_1^2 \frac{\left|\Delta E_1^{(j)}\right| - \left|\Delta E_1^{(j-1)}\right|}{\tilde{\alpha}_i^{(j)} - \tilde{\alpha}_i^{(j-1)}}, \quad \lambda = Const \tag{79}$$

$$\tilde{\beta}_i^{(j+1)} = \tilde{\beta}_i^{(j)} - \lambda_1^2 \frac{\left|\Delta E_1^{(j)}\right| - \left|\Delta E_1^{(j-1)}\right|}{\tilde{\beta}_i^{(j)} - \tilde{\beta}_i^{(j-1)}} \tag{80}$$

It should be noted that the predator cannot rerun the actual trajectory after the pursuit has been ended: he can change $|\Delta E|$ only by rerunning the image of this trajectory, i.e., by simulating the solutions to Eqs. (59) – (61).

Eventually:

$$\lim_{j \to \infty} \tilde{\alpha}_i^{(j)} = \alpha_i, \quad \lim_{j \to \infty} \tilde{\beta}_i^{(j)} = \beta_i \tag{81}$$

i.e., the predator's image of the prey coincides with the prey's self image.

Thus, if Eq. (67) is associated with the innate properties, Eq. (80) can be associated with an acquired, or learned properties.

Obviously the prey learns from the same experience, and as a result, he can acquire some defenses by correcting his image of the predator, i.e., by departing from Eq. (67) to the following:

$$\tilde{p}_i^{(1)} = Sin^2 \sum_{j=1}^{N-1} \left(\tilde{\alpha}_j^{(i)} \tilde{f}_{x_j}^{(1)} + \alpha_1^{(i)} \right), \quad i = 1,2,3; \tag{82}$$

where the weights $\tilde{\alpha}$ and $\tilde{\beta}$ are adjusted by means of minimizing the difference

$$|\Delta E_2| = |E - \tilde{E}_2| \tag{83}$$

8.   **Self-reproduction**

Self-reproduction is one of the privileges of living systems. In order to simulate it within the framework of our phenomenological formalism, we have to make the following assumption: all the species of the same genotype have the same probabilistic invariants of their behaviors. In other words, their trajectories are different samples of the same stochastic process, i.e.they are different on the level of deterministic details, but are identical on the level of statistics. Then the self-production process can be simulated by throwing into the "battlefield" new and new samples of the same stochastic process (for instance, the one described by Eq. (8)). The rate of reproduction has to be governed by the logistic equation:

$$N_{j+1} = N_j - \gamma N_j (1 - N) \tag{84}$$

where N is the population density, γ is the coefficient describing effects of food availability and death rate.

Thus, again we arrive at two different types of descriptions: the global picture is still expressed by Eq. (8), i.e., by the mental dynamics in the probability space; the local

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picture, or motor dynamics represented by a set of nonlinear random walks (7) whose density (in physical space) is expressed by Eq. (84).

If several genotypes occupy the same physical space then the global picture (in the mental space) is represented by Eqs. (60), (61), (63) and (64), while the motor dynamics is described by a two set of random walks (59) and (62) whose densities are expressed by the corresponding versions of Eq. (84).

9. **Collective performance**

In this section we will briefly describe collective phenomena In the proposed model which combine the paradigms discussed above (on the level of individual or pairs of species) with the effects of swarms of species. In other words, we will depart from a single monad and move to a system of interacting (collaborating or competing) monads within the framework of dynamics of intelligent systems.

a. *Collaboration.* Suppose that there are several different, but "friendly" swarms of species having the probability distributions

$$f_j = f_j\left(\{x_j\},\{y_i\},\{z_j\},t\right); j = 1,2,\ldots S \tag{85}$$

where $\{x_j\},\{y_j\}$ and $\{z_j\}$ are sets of space coordinates occupied by the species of the jth swarm.

We will postulate that within the framework of our formalism, the "friendliness" is equivalent to the existence of the joint probability

$$f = f\left(\{x_1\},\ldots\{z_s\}\right) \tag{86}$$

so that f_j in Eq. (84) are interpreted as the conditional probabilities:

$$f_j = f_{\{j\}|\{\ \}}\left(\{x_1\}\ldots\{z_s\}\right) \tag{87}$$

As shown in [8], that imposes upon f_j the following constraints

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$$\frac{\partial^2}{\partial x_j \partial x_k} \ell n \frac{f_j}{f_k} = 0; \quad j,k = 1,2,\ldots s; \quad j \neq k \tag{88}$$

If these constraints are satisfied, one can describe the joint evolution of all the S swarms by only one equation:

$$f_{t+\tau,\{x\}} = \sum_{j=1}^{s} \left[p_j f_{t\{x\}_j - \{h\}j} + (1 - p_i) f_{t,\{x\}_j + \{h\}_j} \right] \tag{89}$$

which represents the evolution of the image of the whole set of species in the probability space. This evolution as a collective brain controls motor dynamics of each specie:

$$x_{t+\tau}^{(i)} = x_t^{(j)} + h_j Sgn(R + \mu_j), \quad j = 1\ldots3 \tag{90}$$

in a centralized way.

The simplest version of the dependence $p_j(f)$ is

$$p_j = Sin^2(\alpha_j f + \beta_j), \quad \mu_j = p_j - \frac{1}{2} \tag{91}$$

which is similar to Eq. (6) for a single specie. However, here the coefficients α_j and β_j may depend upon the population density N, i.e.,

$$\alpha_j = \alpha_j(N), \quad \beta_j = \beta_j(N) \tag{92}$$

At the same time, the coefficient γ in Eq. (84) is likely to depend upon the probability f, i.e.,

$$\gamma = \gamma(f) \tag{93}$$

As a result, Eqs. (92) and (93) couple the motor-mental dynamics (90),(91) with the dynamics of the population density (84).

Hence, in addition to multi-dimensional version of the nonlinear effects discussed for a single specie, such as the spontaneous self-organization, one can expect phenomena

associated with many-body problem: aggregation, formation of new alliances, explosions of the population densities, etc.

b. Competition.. Suppose the swarms described by the probabilities (85) are "hostile." In terms of our formalism it means that the constraints (88) are not satisfied, and therefore, a joint probability (86) does not exist. In other words, the hostile swarms cannot be controlled by a unified "collective brain" as in the previous case. However they can be entangled in a more sophisticated way. Indeed, here instead of Eq (86), one arrives at a set of S coupled equations:

$$f^{(j)}_{t+\tau,\{x\}} = \sum_{i=1}^{3}\left[p_i^{(j)} f^{(j)}_{t,\{x\}_i-\{h\}_i} + \left(1 - p_i^{(j)}\right) f^{(j)}_{t,\{x\}_i+\{h\}_i}\right], \quad j = 1, 2, \ldots s \tag{94}$$

where

$$p_i^{(j)} = p_i^{(j)}\left(f^{(1)}, \ldots f^{(s)}\right) \tag{95}$$

Each of these equations represent the evolution of the image of the corresponding swarm; however, these evolutions are coupled via Eqs. (95).

In order to emphasize the fundamental difference between the unified evolution (89), i.e., the collective brain, and the coupled evolutions (94), one has to recall that in physics the violation of compatibility conditions are usually associated with fundamentally new concepts or a new physical phenomenon. For instance, incompatibility of velocities in a fluid, i.e., non-existence of a velocity potential:

$$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = curl\, v \neq 0 \tag{96}$$

introduces vorticity and rotational flows. In the same way, the violations of Eqs. (88), i.e., non-existence of a joint probability (89), leads to coupled evolution of the stochastic processes (94), while the degree of the incompatibility

$$ink\left(f^{(j)}, f^{(k)}\right) = \frac{\partial^2}{\partial x_j \partial x_k} \ell n \frac{f^{(j)}}{f^{(k)}} \tag{97}$$

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can be interpreted as a some sort of "vorticity" in the probability space.

As mentioned earlier, the "vorticity" makes impossible to find such a transformation of the coordinates x_j which would decouple the stochastic processes (94), i.e., these processes are entangled.

Thus, the "vorticity" (97) brings a new dimension in the complexity of the motor-mental dynamics (89,90): it makes the control of the motor dynamics of each specie less centralized and more distributed. In addition to that, as shown in [8], the information capacity of a set of entangled stochastic processes (94) is greater than that of the processes having the joint probability (89).

In the same way as it was described for a simple specie, the evolutions (85) and (90) can be driven not only by nonlinear instability, but by the objective as well, and that includes learning, self-nonself discrimination (on the level of swarms), calibration, etc.

10.   **Minimum-free-will principle**

In our previous discussion, for the proof of concept, the nonlinear function $p(f)$ has been chosen in the simplest form (6) or (38). However, the only restriction imposed upon this function is the condition

$$0 \leq p \leq 1. \tag{98}$$

Therefore, in general , it can be sought in the form:

$$p = Sin^2[\varphi(f)] \tag{99}$$

where $\varphi(f)$ is an arbitrary function or a functional of f.

If this function is parametrized, for instance, as

$$\varphi(f,\alpha) = \sum_{k=0}^{4} \alpha_k f^k \tag{100}$$

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then the weights α_k can be found from the objective by minimization of the corresponding functional (see Eqs. (41), or (73)-(76).

But suppose that there are several different ways in which the same objective can be achieved, i.e., the function (100) includes a set of weights β

$$\varphi = \varphi(f, \alpha, \beta) \tag{101}$$

which do not affect the objective. How the specie should solve such a redundancy problem?

Let us assume that the physical (i.e., the passive) component of the specie motion is a symmetric random walk which is a discretized version of the Brownian motion.

Then the transitional probability p in Eq. (99) can be decomposed as:

$$p = Sin^2\left[\hat{\varphi}(f) + \frac{\pi}{4}\right], \quad i.e., p = \frac{1}{2} \quad if \quad \hat{\varphi}(f) \equiv 0 \tag{102}$$

In this form, the nonlinear component of p, i.e., the function $\hat{\varphi}(f)$ represents the deviation from the passive motion, i.e., the "free will."

Now we will make the following statement: if a specie can achieve its objective by several different ways, it will choose one which minimizes the deviation from the passive motion, i.e., it will minimize its free will component. In other words, if a specie is offered a "free ride" by physics, it should take it. This minimum-free-will principle can be associated with the Gauss minimum constraints principle according to which the motion of a constrained system minimizes the deviation from the corresponding free motion. However, in contradistinction to that, the minimum-free-will principle is not required by physics, but it is rather imposed by biology. Indeed, a "crazy" specie can move "against" the minimum-free-will principle, but it will waste its energy and "intellectual" effort, and as a result, its chances for survival will be decreased.

A natural measure of deviation from the passive motion is the difference:

$$|\Delta H| = |H_0 - H| \tag{103}$$

where H_0 and H are the entropies of the passive and the actual motions, respectively.

Now we can give the mathematical formulation of the minimum-free-will principle: if the specie objective is defined in the time interval $0 \le t \le T$, its motion will minimize the free-will measure

$$F_w = \sum_{t=0}^{T} |H_0 - H|_\tau \qquad (104)$$

subject to the objective. In other words, if the weights α in Eq. (84) are defined by the objective, then the redundant weights β^* must be found from the condition:

$$F_w(\beta^*) = \inf F(\beta) \qquad (105)$$

One should recall that

$$H = -\sum_{x=0}^{L} f_x \, log_2 f_x, \quad 0 \le x \le L$$

and f is found from Eq. (2) where $p = Sin^2[\varphi(f, \beta)]$

11. **Reflexive Chains**

As follows from the previous sections, the fundamental difference between a physical and an intelligent system is the possession of the self-image by the latter. In section 7 we have introduced a set of images: the self-image and the image of the adversary. In this section we will describe a general picture of the world of images, and in particular, the complexity of its reflexive structures.

The notion of reflexive structures was introduced in mathematical psychology [11] based upon an axiomatically defined formalism. In our approach, the same phenomenon is represented as an interaction between a stochastic process (describing the motor dynamics, i.e., the actual reality) and its own probability evolution (describing the mental dynamics, i.e., the self-image); no additional postulates are required. However, the feedback which controls the motor dynamics, includes a set of the control parameters α_i

(see, for example, Eq. (38)) which are supposed to be found from the objective (see Eqs. (42) and (43)). And only when the same objective can be achieved by several different combinations of the control parameters, i.e., when these parameters are redundant, the minimum-free-will principle has to be applied.

Let us consider a set of n interacting monads. Each monad is characterized by its motor coordinates $x_j(t)$ and the self-image coordinates $f_j(t.x_j)$ $j = 1, 2, \ldots n$. Such a state of the system we will associate with the first level of reflexion. On the second level of reflection, each monad has images of the rest of monads, i.e., $f_{jk}(t, x_k)$, $k \neq j$. On the third level of reflexion, each monad has images of images of the rest monads of themselves as well as of others, i.e., $f_{jk\ell}(t, x_k, x_s), k \neq j$, etc.

It can easily be verified that the number of images on the first level of reflexion

$$\ell_1 = n$$

on the second level of reflexion

$$\ell_2 = n(n-1) = n(\ell_1 - 1) \quad \text{etc.}$$

Finally

$$
\begin{aligned}
\ell_m &= n(\ell_{m-1} - 1) = \\
&= n + n(n-1) + n[n(n-1) - 1] + \ldots etc. \propto n^m
\end{aligned}
\tag{106}
$$

where m is the level of reflexion.

Thus, the number of images characterizing the state of an n-monad system rows polynomially as a function of n, and exponentially as a function of m.

For a m-level of reflexion, the behavior of each monad is described by a system of $\dfrac{\ell_m}{n} + 1$ equations corresponding to the variables $x_j, f_j, f_{jk}, f_{jks}, f_{jksq}$, etc. The first group of variables, x_j, characterizing the motor dynamics, are governed by the equations of the type (59); the second group of variables, f_j, characterizing the dynamics of the self-image, are governed by the equations of the type (60); the third, group of variables,

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f_{jk}, characterizing the dynamics of the images of the other monads, are governed by the equations of the type (61), etc. All these equations are coupled via the common objective of the type (68) which adds another set of dynamical equations of the type of (73)-(76) governing the adjustments of the control parameters α_i, β_i, etc.

As follows from Eq. (106), the world of images is never complete: each new level of reflexation brings in an additional set of images with the corresponding number of the governing dynamical equations, and that leads to deeper and deeper interactions between monads. It may happen that some monads have longer chains of images than others; obviously, in case of competition, these monads can better predict the evolution of the whole system, and that will give them advantage over those with the shorter chains of images.

Let us assume now that the number of the levels of reflexion tends to infinity, i.e., $m \to \infty$, and, as follows from Eq. (106), the number of interacting images ℓ_m as well as the number of the correspondings governing equations, grow exponentially. Does the system's behavior tend to some limit pattern such that starting with some larger m it does not change any more? Even without rigorous mathematical analysis it is obvious that the answer to this question depends upon the structure of the objective. Indeed, suppose that the system consists of n monads, and let us start with the case when they are collaborating. Then their objective can be formulated in terms of minimization of some functional E of the coordinates $x_j(t), f_j(t, x_j)$ etc., through the control parameters α_i under the assumption that this functional has a unique global minimum. Since each new level m of reflexion brings in a new set of the control parameters, obviously

$$E_{m+1} \le E_m, \text{ and } E_m \to \min \text{ at } m \to \infty \tag{107}$$

This means that the system is attracted to a certain pattern of behavior when the number of the reflexion levels is sufficiently large, and therefore, its complexity is limited.

If the functional E has several local minima, then each of them can be approached with some probability, i.e., the system may have several different limit patterns of behavior if $m \to \infty$.

Finally, for competing monads, the functional E may have only saddle points when a minimum with respect to one set of coordinates corresponds to a maximum with respect to another. In this situation, the system does not have any stable limit behavior, and it will endlessly increase its complexity as $m \to \infty$. But does such an unstable limit behavior have some stable invariants at $m \to \infty$ in the same way as chaos does? At this stage we do not have an answer to that question.

12. Discussion and conclusion

There were many attempts to describe the behavior of living systems by the mathematical formalism of classical physics which includes Newtonian mechanics, thermodynamics and statistical mechanics, [1]. Not withstanding indisputable success of this approach, we will concentrate our attention to its limitations. In order to illustrate that, we will start our discussion with the following example: consider a small physical particle in a state of random migration due to thermal energy, and compare its diffusion, or physical random walk, with a biological random walk performed by a wild-type bacterium which can be associated with the simplest biological particle, i.e., a monad. The fundamental difference between these two types of motions can be detected in the probability space: the probability evolution of the physical particle (which can be associated with the Fokker-Planck equation) is always linear, and it has only one attractor – a stationary stochastic process where the motion is trapped. On the contrary, a typical probability evolution of a biological particle is nonlinear: it can have many different

attractors, but eventually each attractor can be departed without any "help" from outside. However, such a behavior violates the second law of thermodynamics unless there is another "hidden" object which interacts with the monad. In order to find this object, one has to turn to mathematical psychology which postulates that a human possesses a self-image and interacts with it. (Actually the concept of self-image was introduced by Kant, and recently Caltech reported a discovery of specific parts of a human brain responsible for the self-awareness, Caltech News, 1999). In terms of the mathematical formalism of classical physics, the self-image can be represented by the probability evolution (mental dynamics) associated with the corresponding random walk (motor dynamics). Then the interaction between the monad and its self-image is implemented by the feedback from mental to motor dynamics, and that makes the probability evolution nonlinear. Actually the deviation from linear evolution expresses the "free-will" of the monad. From the physical viewpoint, the self-image is an external object, and that reconciles biological random walk with the second law of thermodynamics. (The need for a reconcilation was expressed by E. Schrödinger in his book "what is life?", 1944).

Thus, it has been proposed that in order to capture dynamical invariants of behavior of living system, classical physics should be equipped with an additional postulate, namely, that each living system possesses the self-image. This self image can be incorporated into the mathematical formalism of nonlinear dynamic which evolve in the probability space. The only difference between the classical and the probabilistic nonlinear dynamics is in additional constraints imposed upon the latter by the normalization conditions followed from the definition of probability. Actually the self-image postulate can serve as a definition of living systems characterized by purposeful movements.

It has been demonstrated that within the formalism introduced above, a living system can predict future in terms of probabilities due to smoothness of evolution in the

probability space (such a smoothness does not exist in actual space because of irregularities of a random walk). This ability which increases chances for survival can be considered as a basic component of intelligence.

The proposed model of the simplest biological particle, monad, consists of a generator of stochastic processes which represents the motor dynamics in the form of nonlinear random walks, and a simulator of the nonlinear version of the Fokker-Planck equation which represents the mental dynamics. Both components can be implemented by physical hardware (neural networks, cellular automation, etc.), and thereby, one can introduce artificial intelligent systems which have the same phenomenology as natural ones.

There have been shown that coupled motor-mental dynamics can simulate such processes as emerging self-organization, decision-making based upon "common sense," predator-prey evolutionary games, collective brain, etc. Therefore, the proposed model can serve as a starting point for a unified approach to model behavior of intelligent systems.

Appendix

The model of motor dynamics (see Eqs. (2), (7), (15), etc.) has been presented in the form of a discrete automation since it was assumed that at any given moment the system stays in one of the few discrete states while the transition between such states was neglected.

In this Appendix we will describe the complete physical scenario which includes both discrete (probabilistic) and continuous (deterministic) components of the motor dynamics thereby implementing Eqs. (2), (7), (15), etc. by "physical hardware," i.e., without random number generators. The approach is based upon the non-Lipschitz

dynamics [9,10] which introduces discreteness and randomness in a natural way, i.e., without man made devices.

Let us consider a non-dimensional motion of a particle of unit mass driven by a non-Lipschitz force:

$$\dot{v} = v v^{1/3} \sin \omega t, \quad v = const, \qquad [v] = \frac{m^{2/3}}{\sec^{5/3}} \tag{A1}$$

$$\dot{x} = v \tag{A2}$$

where v and x are the particle velocity and position, respectively.

Subject to the zero initial condition

$$v = 0 \qquad \text{at} \qquad t = 0 \tag{A3}$$

equation (A1) has a singular solution

$$v = 0 \tag{A4}$$

and a regular solution

$$v = \pm\left(\frac{4v}{3\omega}\sin^2\frac{\omega}{2}t\right)^{3/2} \tag{A5}$$

These two solutions coexist at t = 0, and this is possible because at this point the Lipschitz condition fails:

$$\left|\frac{\partial \dot{v}}{\partial v}\right|_{t\to 0} = \frac{1}{3}v v^{-2/3}\sin \omega t\big|_{t\to 0} \to \infty \tag{A6}$$

Since

$$\frac{\partial \dot{v}}{\partial v} \rangle 0 \qquad \text{at} \quad |v| \neq 0, \qquad t \rangle 0 \tag{A7}$$

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the singular solution (A4) is unstable, and the particle departs from rest following the solution (A5). This solution has two (positive and negative) branches (since the power in (A5) includes the square root), and each branch can be chosen with the probability p and (1-p) respectively. It should be noticed that as a result of (A5), the motion of the particle can be initiated by infinitesimal disturbances (such motion never can occur when the Lipschitz condition holds: an infinitesimal initial disturbance cannot become finite in finite time[8,9]).

Strictly speaking, the solution (A5) is valid only in the time interval

$$0 \le t \le \frac{2\pi}{\omega} \tag{A8}$$

and at $t \le 2\pi / \omega$ it coincides with the singular solution (A4)

For $t \rangle 2\pi / \omega$ equation (A4) becomes unstable, and the motion repeats itself to the accuracy of the sign in equation (A5).

Hence, the particle velocity υ performs oscillations with respect to its zero value in such a way that the positive and negative branches of the solution (A5) alternate randomly after each period equal to $2\pi / \omega$.

Turning to equation (A2), one obtains the distance between two adjacent equilibrium position of the particle:

$$\Delta x_i = x_i - x_{i-1} = \pm \int_0^{2\pi/\omega} \left(\frac{4v}{3\omega} \sin \frac{\omega}{2} t \right)^{3/2} dt = 64(3\omega)^{-5/2} v^{3/2} = \pm h \tag{A9}$$

Thus, the equilibrium positions of the particle are

$$x_0 = 0, \qquad x_1 = \pm h, \qquad x_2 = \pm h \pm h... \tag{A10}$$

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while the positive and negative signs randomly alternate with probabilities p and (1-p), respectively.

Obviously, the particle performs an unrestricted random walk: after each time period

$$\pi = \frac{2\pi}{\omega}$$

(A11)

it changes its value on $\pm h$ [see equation (A10)].

The probability density f(x,t) is governed by the following difference equation:

$$f(x,\, t\, +\, \tau) = pf(x\, -\, h, t) + (1 - p) f(x\, +\, h, t)$$

(A12)

which represents a discrete version of the Fokker-Planck equation, while

$$\int_{-\infty}^{\infty} f(x,t)\, dx = 1$$

(A13)

Several comments to the model (A1) and its solution have to be made.

Firstly, the "viscous" force

$$F = -v\upsilon^{1/3}$$

(A14)

includes static friction (see Eq. A6) which actually causes failure of the Lipschitz condition. These type of forces are well-known in theory of visco-plasticity [12]. It should be noticed that the power $1/3$ can be replaced by any power of the type:

$$k = \frac{2n - 1}{2n + 1}, \quad n = 1,2,... \text{ etc}$$

(A15)

with the same final result (A12). In particular, by selecting large n, one can make k close to 1, so that the force (A13) will be almost identical to its classical counterpart

$$F_c = -v\upsilon$$

(A16)

everywhere excluding a small neighborhood of the equilibrium point $\upsilon = 0$, while at this point

$$\frac{dF}{d\upsilon} \to \infty, \quad \text{but} \left|\frac{\partial F_c}{\partial \upsilon}\right| \to 0 \quad \text{at} \quad \upsilon \to 0 \tag{A17}$$

Secondly, without the failure of the Lipschitz condition (A6), the solution to Eq. (A1) could not approach its equilibrium $\upsilon = 0$ in finite time, and therefore, the paradigm leading to random walk (A12) would not be possible.

Finally, we have to discuss the infinitesimal disturbances mentioned in connection with the instability of the solutions (A5) at $\upsilon = 0$. Actually the original equation should be written in the form:

$$\dot{\upsilon} = \nu\upsilon^{1/3} \sin\omega t + \varepsilon, \ \varepsilon \to 0 \tag{A18}$$

where ε represents infinitesimal disturbances. It should be emphasized that this process is not driving the solution of Eq. (A18): it only triggers the mechanism of instability which controls the energy supply via the harmonic oscillations $\sin\omega t$. As follows from Eq. (A18), the disturbance ε can be ignored when $\dot{\upsilon} = 0$ or when $\dot{\upsilon} \neq 0$, but the equation is stable, i.e. $\upsilon = \pi\omega, 2\pi\omega,....$etc. However, it becomes significant during the instants of instability when $\dot{\upsilon} = 0$ at $t = 0, \pi/2\omega$ etc. Indeed, at these instants, the solution to Eq. (A18) has a choice to be positive or negative if $\varepsilon = 0$, (see Eq. (A5)). However, with $\varepsilon \neq 0$,

$$sign\ \upsilon = sign\ \varepsilon \ \text{at}\ t = 0,\ \pi/2\omega,...\text{etc}.... \tag{A19}$$

i.e., the sign of ε at the critical instances of time (A19) uniquely defines the evolution of the dynamical system (A18). Thus, the dynamical system (A18) creates a binary time series which, in turn, generates a random-walk-paradigm [equation (A2)].

We will start with the case when

$$\varepsilon = \varepsilon_0 Sin\frac{1}{\upsilon}, \ \varepsilon_0 \to 0. \tag{A20}$$

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The function $Sin\dfrac{1}{\upsilon}$ is well-known in classical calculus: it oscillates about zero with unbounded frequency at $\upsilon \rightarrow 0$, and therefore, with equal probability it can be positive or negative at $\upsilon \rightarrow 0$ if the precision of its representation is finite.

Therefore, the statistical signature of the random walk described by Eqs. (A18), (A2), and (A20) is expressed by the solution to equation (A12) and equation (A13) at p=1/2. With the initial conditions

$$f(0,0) = 1, \quad f(x,0) = 0 \quad \text{if} \quad x \neq 0$$

it is a symmetric unrestricted random walk:

$$f(x,t) = C_n^m 2^{-n}; \quad m = \frac{1}{2}(n+x); \quad n = \text{integer}\left(\frac{2\omega t}{\pi}\right) \tag{A21}$$

Here the binomial coefficient should be interpreted as 0 whenever m is not an integer in the interval $[0,n]$ and n is the total number of steps.

One can verify (by substitution) that the function $\omega = Sin\dfrac{1}{\upsilon}$ is the solution to the following differential equation:

$$\frac{d\omega}{d\upsilon} + \frac{1}{\upsilon^2}\sqrt{1-\omega^2} = 0 \quad or \quad \dot{\omega} = -\frac{\dot{\upsilon}}{\upsilon^2}\sqrt{1-\omega^2}$$

$$\tag{A22}$$

$$\dot{\omega} = -\frac{\dot{\upsilon}}{\upsilon^2}\sqrt{1-\omega^2}$$

Both equations in (A22) suffer from a failure of the Lipschitz conditions at $\upsilon = 0$.

Thus, the probabilities described by equation (A12) are simulated by the dynamical system (A22) and (A2) without an explicit source of stochasticity (while the 'hidden' source of stochasticity is in finite precision of the functions representation combined with the non-Lipschitz instability).

Combining several dynamical systems of the type (A22) and (A2) and applying an appropriate change of variables, one can simulate a probabilistic Turing machine which transfers one state to another with a prescribed transitional probability. Non-

Markovian properties of such a machine can be incorporated by introducing time-delay terms in equation (A2):

$$\dot{x} = \upsilon(t) + \alpha_1 \upsilon(t - \tau_0) + \alpha_2 \upsilon(t - 2\tau_0) + \dots \tag{A23}$$

However, there is a more interesting way to enhance the dynamical complexity of the system (A22) and (A2). Indeed, let us turn to Eq. (A23) and introduce a feedback from Eq. (A2) to Eq. (A22) as follows:

$$\varepsilon = \varepsilon_0(\omega - x) \tag{A24}$$

Then the number of negative (positive) signs in the string (25) will prevail if $x \rangle 0 \, (x \langle 0)$ since the effective zero-crossing line moves down (up) away from the middle. Thus, when $(x = 0)$ at $t = 0$, the system starts with an unrestricted random walk as described above, and $|x|$ grows. However, this growth changes signs in Eq. (A24) such that $\dot{x} \langle 0$ if $x \rangle 0$, and $\dot{x} \rangle 0$ if $x \langle 0$. As a result of that

$$x_{max} \leq y_{max}, \quad x_{min} \geq y_{min} \tag{A25}$$

where y_{max} and y_{min} are the largest and the smallest values in the time series y (t), respectively. Hence, the dynamical system (A22), and (A2) simulates a restricted random walk with the boundaries (A25) implemented by the dynamical feedback (A24), while the probability

$$p(sign\varepsilon \rangle 0) = \begin{cases} 0 \text{ if x } \geq y_{max} \\ 1 \text{ if x } \leq y_{min} \end{cases} \tag{A26}$$

For the sake of qualitative discussion, assume that p change linearly between $x = y_{min}$ and $x = y_{max}$, i.e.,

$$p = \begin{cases} 0 & \text{if } x > y_{max} \\ \dfrac{y_{max} - x}{y_{max} - y_{min}} & \text{if } y_{min} \leq x \leq y_{max} \\ 1 & \text{if } x < y_{min} \end{cases} \tag{A27}$$

Then the simulated restricted random walk is a solution to equations (A12) and A27). An alternative approach to representation of the bias ε in Eq. (A18) was described in [9] where ε was sampled from the chaotic time series of the logistic map.

Acknowledgements

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